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Energy spectra for non-linear oscillators with broken symmetry

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Abstract. The statistical properties of energy spectra of a classical chaotic system are investigated, in particular for the effect of broken symmetry. Small symmetry breaking terms produce significant deviations from a Wigner distribution and also characteristic differences to the Δ_3 function of the Gaussian orthogonal ensemble. Furthermore, a saturation effect for Δ_3 was found as predicted in the literature.

Recently there has been growing interest in the quantum manifestations of classically regular and chaotic dynamical behaviour. In many investigations (Berry and Tabor 1977, Berry and Robnik 1984, Berry 1983, 1985, Bohigas *et al* 1984a, b, Bohigas and Giannoni 1984, Casati *et al* 1985, Haller *et al* 1984, McDonald and Kaufmann 1979, Pullen and Edmonds 1981, Seligman *et al* 1984, Seligman and Verbaarschot 1985) it has been found that the energy spectrum of a quantum system shows different behaviour for classically integrable and non-integrable systems in the semiclassical limit. Applying a semiclassical quantisation method Berry and Tabor (1977) were able to show that the nearest-neighbour spacing (NNS) distribution of the energy eigenvalues of an integrable system is given by a Poisson distribution. For a non-integrable system, however, the eigenvalues are distributed rather regularly and the NNS distribution agrees well with a Wigner distribution. The fluctuation properties of these systems coincide remarkably well with the fluctuation properties of the Gaussian orthogonal ensemble (GOE) if the system has time-reversal symmetry. But it is important when looking for such behaviour that one classifies the energy eigenvalues according to the symmetries of the underlying system. Only those eigenvalues which belong to the same symmetry class show a Wigner distribution and the fluctuation properties of the GOE. If one mixes the levels of different symmetries one gets a superposition of uncorrelated GOE spectra (Bohigas *et al* 1984a, b) unless there are degeneracies implied by symmetry. A frequently used measure for the fluctuations of the eigenvalue sequences is the Δ_3 function of Dyson and Mehta (1963). Berry (1985) was able to calculate this function for regular and chaotic systems. For both types of systems there exists a maximal value L_{max} such that the Δ_3 function over a range of L mean level spacings with $1 \leqslant L \leqslant L_{max}$ is given by $(L/15)$ for regular systems and $((1/\pi^2)\ln L + \text{constant})$ for chaotic systems. For values of L greater than L_{max} the Δ_3 function saturates non-universally at a value determined by short classical periodic orbits. For integrable systems this saturation has been observed by Seligman *et al* (1984) and Casati *et al* (1985). In this paper we would like to study the influence of small symmetry breaking terms on the energy level distribution of a system whose classical behaviour is almost

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totally chaotic. It is expected that this symmetry breaking term will superpose different level distributions of the original unperturbed Hamiltonian. Since, however, the symmetry breaking term does not change the classical chaotic behaviour of the system, we expect a transition to a single GOE as the symmetry breaking term gets stronger. Another effect of symmetry breaking has been investigated by Bohigas *et al* (1986), who studied the transition from GOE to GUE by breaking the time-reversal symmetry.

The starting point of our investigation is a system well known in the literature (Pullen and Edmonds 1981, Haller *et al* 1984) consisting of two oscillators with a quartic coupling term. The Hamiltonian for this system can be written as

$$H_0 = \tfrac{1}{2}p_1^2 + \tfrac{1}{2}p_2^2 + \tfrac{1}{2}q_1^2 + \tfrac{1}{2}q_2^2 + 4\kappa q_1^2 q_2^2. \tag{1}$$

Its symmetry group is denoted by C_{4v} and the eigenfunctions can be divided into five symmetry classes indicated by A_1, A_2, B_1, B_2 and $\tilde{E}$. The eigenvalues in class $\tilde{E}$ are twofold degenerate. The classical behaviour of (1) depends only on the product of coupling strength κ and energy E; the system shows a transition from regular to chaotic motion in the interval $0.2 \leqslant \kappa E \leqslant 0.6$. The NNS distribution of the corresponding quantum system exhibits a continuous transition from a Poisson to a Wigner distribution with increasing energy, which reflects the transition from an integrable to a non-integrable behaviour in the classical system.

A symmetry breaking term which destroys the C_{4v} symmetry can be chosen to be linear in both coordinates and has the form

$$H_1 = \mu(q_1 - q_2). \tag{2}$$

Applying the canonical transformation $p_x = (p_1 - p_2)/\sqrt{2}$, $p_y = (p_1 + p_2)/\sqrt{2}$, $x = (q_1 - q_2)/\sqrt{2}$, $y = (q_1 + q_2)/\sqrt{2}$ the total Hamiltonian $H_0 + H_1$ is transformed to

$$H = \tfrac{1}{2}p_x^2 + \tfrac{1}{2}p_y^2 + \tfrac{1}{2}x^2 + \tfrac{1}{2}y^2 + \kappa(x^2 - y^2)^2 + \sqrt{2}\mu x. \tag{3}$$

The only symmetry operation that leaves H invariant is $y \to -y$. For a fixed coupling strength, $\kappa = 0.1$, the classical behaviour was studied at the energy $E = 10$ as a function of the symmetry breaking term μ. Calculating the dynamical behaviour of phase space distances as a function of time for a large number of randomly chosen points on the energy surface and from inspecting Poincaré sections, we concluded that almost all the available phase space (more than 99%) was filled by chaotic trajectories for $\mu \in [0, 2]$. This was to be expected because system (1) with $\kappa = 0.1$ shows a transition from regular to chaotic dynamics in the energy range $2 < E < 6$.

In the quantum mechanical calculations we first of all study the Hamiltonian (3) in the limit of very small μ. The eigenfunctions of the Schrödinger equation can be divided into two symmetry classes, with eigenfunction even or odd in y. A possible basis for the first symmetry class consists of the eigenfunctions of system (1) which transform themselves according to A_1, B_1 and those eigenfunctions of $\tilde{E}$ which are symmetric in y. The basis of the second class consists of eigenfunctions of A_2, B_2 and those of E which are odd in y. In order to compare the numerical results discussed below with analytic distributions we assume that the energy levels of H_0 are only slightly disturbed by H_1. In this case one can calculate the NNS distribution and the fluctuation properties. Because we are in an energy regime where the eigenvalues of (1) show the properties of GOE spectra, we furthermore assume that the energy spectra of system (3) resemble the superposition of three GOE spectra with weights $\rho_1 = \rho_2 = \frac{1}{4}$ (=weight of A_1, B_1) and $\rho_3 = \frac{1}{2}$ (=weight of $\tilde{E}$). The NNS distribution of such a

superposition is given by Berry and Robnik (1984):

$$P(s) = \frac{d^2}{ds^2} \prod_{i=1}^{3} \operatorname{erfc}(\sqrt{\pi}\rho_i s/2) \tag{4}$$

with the well known error function $\operatorname{erfc}(x)$.

As a measure of the fluctuations in an interval of length L of the unfolded spectrum one commonly uses the Δ_3 function of Dyson and Mehta, which in this case takes the form (Seligman and Verbaarschot 1985)

$$\Delta_{3,\mathrm{mix}}(L) = \sum_{i=1}^{3} \Delta_{3,\mathrm{GOE}}(\rho_i L) \tag{5}$$

where $\Delta_{3,\mathrm{GOE}}(x)$ is the Δ_3 function for the GOE. This procedure will be true only if the perturbation of the energy eigenvalues by the term μx is very small. As the symmetry breaking gets stronger we expect a transition of the spectral properties to those of a single GOE spectrum.

In order to test these expectations we determined the eigenvalues of the Schrödinger equation for system (3) numerically. As a basis for the eigenfunctions of the two symmetry classes we used appropriately displaced harmonic oscillator functions and ordered these functions in such a way so as to obtain a matrix with small bandwidth. The dimension of our matrix was 4000, from which we got about 1000 well converged eigenvalues.

Next we had to unfold the spectra in order to get rid of the variation of the density of energy levels. This was done as described by Bohigas and Giannoni (1984). One determines the average number of levels up to the energy E, $N_{\mathrm{av}}(E)$ and replaces the levels E_i by $\varepsilon_i = N_{\mathrm{av}}(E_i)$. From the sequence $\{\varepsilon_i\}$, which has constant NNS in the mean, we calculate the histograms for the NNS and the Δ_3 function.

We first discuss the behaviour of the NNS distribution as a function of μ. Note that the classical fraction of phase space filled by chaotic trajectories in the energy range discussed below is always greater than 0.99. In figure 1(a) the case for $\mu = 0$ is shown, and one can see that the histogram obtained from the numerically determined spectrum agrees very well with the theoretical result for three superposed uncorrelated GOE spectra with appropriate weights. As μ increases the first significant deviations from this behaviour occur for $\mu \geqslant 0.02$ (in the energy range considered here). In figure 1(b) we show the example $\mu = 0.03$ where the most obvious differences appear at very small values of S, which indicates the level repulsion property of the spectrum. For $\mu \geqslant 0.02$ the NNS distribution could be fitted very well by the Brody distribution (Brody 1973) which interpolates by an additional parameter q between a Poisson ($q = 0$) and a Wigner distribution ($q = 1$). This additional parameter has no direct physical meaning but gives a better fit to the numerical spectrum compared to the formula of Berry and Robnik (1984), especially for small level spacings (Wintgen and Friedrich 1987). The parameter q which gives the best fit depends on μ and on the energy range considered. For $\mu = 0.03$ we find $q = 0.48$ (figure 1(b)), while for $\mu \geqslant 0.3$ one has $q \simeq 1$, and the histogram agrees almost perfectly with a Wigner distribution (see figure 1(c) for $\mu = 1.0$). This behaviour does not change as μ increases still further. The results for the Δ_3 function are given in figure 2. For $\mu = 0$ the theoretical (equation (5)) and numerical results agree up to $L \simeq 20$, but for larger L the Δ_3 function determined from the spectrum lies significantly lower than expected. This indicates that the level fluctuations are suppressed for large values of L. This can already be seen in the unperturbed system (1) for which the energy levels were calculated for different

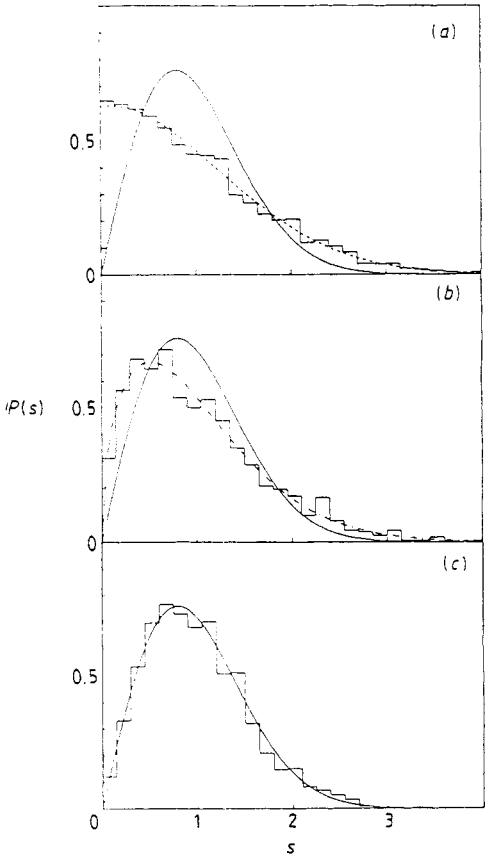

Figure 1. Nearest-neighbour spacing distribution $P(s)$ for system (3) with $\kappa = 0.1$ calculated from levels in the energy range (30-100) for (*a*) ($\mu = 0$), (70-110) for (*b*) ($\mu = 0.03$) and (30-100) for (*c*) ($\mu = 1.0$), using levels with positive and negative parity. The full curve gives a Wigner distribution, the broken curve a superposition of three Wigner distributions with weights $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ and the chain curve shows the Brody distribution with $q = 0.48$.

symmetry classes and found also to saturate in the Δ_3 function. A similar result has been found by Zimmermann *et al* (1987). The saturation in the Δ_3 function was also predicted by Berry (1985) using semiclassical methods. Therefore equation (5), which does not take into account this saturation effect, is valid only up to some $L_{\max}$. As μ increases, the Δ_3 function decreases and its values are between a superposition of three GOE and a single GOE. For $\mu \geqslant 0.02$ the Δ_3 function agrees very well with the expression

$$\Delta_3(L) = \Delta_{3,\mathrm{GOE}}(qL)/q \tag{6}$$

where q is the Brody parameter determined by fitting the NNS. In figure 3 we show the result for the Δ_3 function, which was calculated for $\mu = 0.03$ and 0.1 from the eigenvalues in the energy range $70 \leqslant E \leqslant 110$.

For $\mu \geqslant 0.3$ and $L \leqslant L_{\max}$ the agreement with a single GOE is remarkably good. With increasing μ the value $L_{\max}$ decreases and the Δ_3 function saturates earlier (see figure 2 for $\mu = 1.0$).

In conclusion we have shown that a symmetry breaking term in a classically chaotic system may change drastically the NNS distribution and the fluctuation properties of

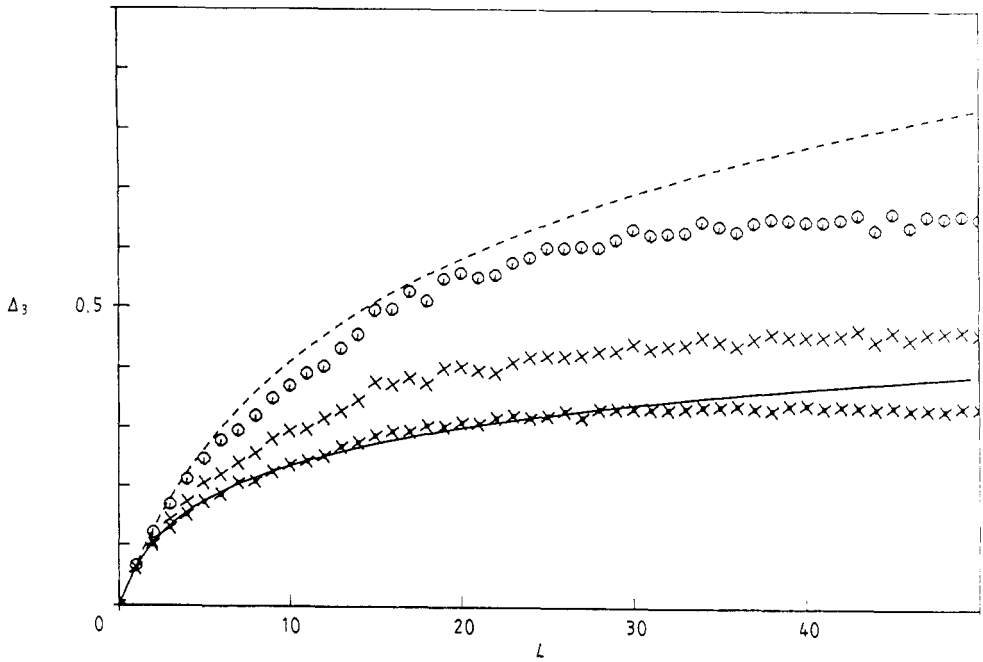

Figure 2. Δ_3 function for system (3) with energy range (70-110), for $\mu = 0$ (○), $\mu = 0.1$ (×) and $\mu = 1.0$ (×), using only levels with positive parity. The full curve represents the GOE result and the broken curve a superposition of three GOE.

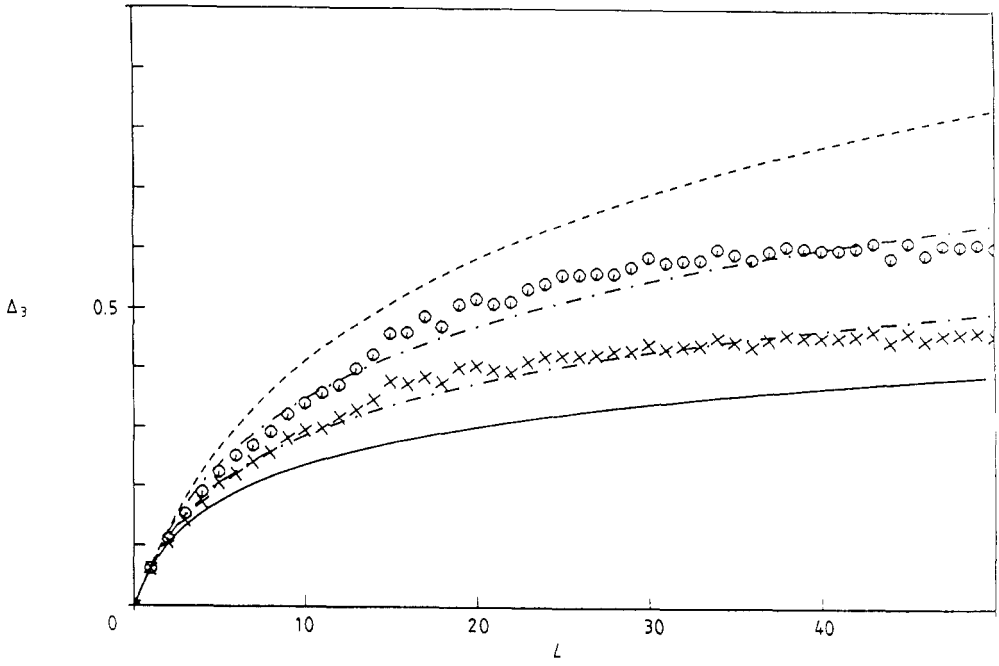

Figure 3. The same as figure 2. ○ shows the case $\mu = 0.03$ and × shows $\mu = 0.1$. The chain curve shows the function (6), where the Brody parameter was determined by fitting the corresponding histograms ($q = 0.50$ for $\mu = 0.03$ and $q = 0.72$ for $\mu = 0.1$).

the corresponding quantum system. These changes have no counterpart in the classical system. As the symmetry breaking gets stronger the spectrum rearranges and one finds a Wigner distribution for the NNS as well as the fluctuation properties of the GOE. For large values of L, however, the Δ_3 function saturates as theoretically predicted (Berry 1985).

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